

MOORE'S CONJECTURE FOR POLYHEDRAL PRODUCTS

YANLONG HAO*, QIANWEN SUN*, AND STEPHEN THERIAULT

ABSTRACT. Moore's Conjecture is shown to hold for generalized moment-angle complexes and a criterion is proved that determines when a polyhedral product is elliptic or hyperbolic.

1. INTRODUCTION

Moore's Conjecture envisions a deep relationship between the rational and torsion homotopy groups of finite CW -complexes. Let X be a finite CW -complex. The *homotopy exponent* of X at a prime p is the least power of p that annihilates the p -torsion in the homotopy groups of X . The space X is *elliptic* if it has finitely many rational homotopy groups and *hyperbolic* if it has infinitely many rational homotopy groups.

Moore's Conjecture: Let X be a finite CW -complex. Then the following are equivalent:

- (a) X is elliptic;
- (b) X has a finite homotopy exponent at every prime p ;
- (c) X has a finite homotopy exponent at some prime p .

The conjecture posits that the nature of the rational homotopy groups should have a profound impact on the nature of the, seemingly unrelated, torsion homotopy groups, and that torsion behaviour at one prime has a profound impact on torsion behaviour at all primes. The conjecture has been shown to hold in a number of cases. Elliptic spaces with finite exponents at all primes include spheres [12, 20], finite H -spaces [13], H -spaces with finitely generated cohomology [4], and odd primary Moore spaces [15]. Hyperbolic spaces with no exponent at any prime include wedges of simply-connected spheres and most torsion-free two-cell complexes [16], and torsion-free suspensions [18]. There are also partial results. In [14] it was shown that if X is elliptic then it has an exponent at all but finitely many primes, and in [19] it was shown that if X is hyperbolic and $H_*(\Omega X; \mathbb{Z})$ is p -torsion free then, provided p is large enough, X has no exponent at p .

Moore's conjecture is also related to an important phenomenon in rational homotopy theory. Félix, Halperin and Thomas [7] proved the remarkable fact that a finite CW -complex is either

2010 *Mathematics Subject Classification.* Primary 55Q05, Secondary 13F55, 55P62, 55U10.

Key words and phrases. Moore's conjecture, elliptic, hyperbolic, polyhedral product.

* Research supported by the National Natural Science Foundation of China (No. 11571186).

elliptic or its total number of rational homotopy groups below dimension n grows exponentially with n . There is no hyperbolic space whose rational homotopy groups have polynomial growth.

In this paper we consider Moore's conjecture, and the notions of being elliptic or hyperbolic, in the context of polyhedral products. Let K be a simplicial complex on m vertices. For $1 \leq i \leq m$, let (X_i, A_i) be a pair of pointed CW -complexes, where A_i is a pointed subspace of X_i . Let $(\underline{X}, \underline{A}) = \{(X_i, A_i)\}_{i=1}^m$ be the sequence of CW -pairs. For each simplex (face) $\sigma \in K$, let $(\underline{X}, \underline{A})^\sigma$ be the subspace of $\prod_{i=1}^m X_i$ defined by

$$(\underline{X}, \underline{A})^\sigma = \prod_{i=1}^m \overline{X}_i \quad \text{where} \quad \overline{X}_i = \begin{cases} X_i & \text{if } i \in \sigma \\ A_i & \text{if } i \notin \sigma. \end{cases}$$

The *polyhedral product* determined by $(\underline{X}, \underline{A})$ and K is

$$(\underline{X}, \underline{A})^K = \bigcup_{\sigma \in K} (\underline{X}, \underline{A})^\sigma \subseteq \prod_{i=1}^m X_i.$$

The topology of polyhedral products has received a great deal of attention recently due to their central role in toric topology [1, 3, 9, 10, 11]. Important special cases include *moment-angle complexes* \mathcal{Z}_K , when each pair (X_i, A_i) equals (D^2, S^1) , and *generalized moment-angle complexes* $\mathcal{Z}_K(D^n, S^{n-1})$, when each pair (X_i, A_i) equals (D^n, S^{n-1}) .

To state our results some definitions are needed. Write $[m]$ for the vertex set $\{1, \dots, m\}$. Let Δ^{m-1} be the standard m -simplex with vertex set $[m]$. The faces of Δ^{m-1} can be identified with sequences (i_1, \dots, i_k) for $1 \leq i_1 < \dots < i_k \leq m$. If K is a simplicial complex on the vertex set $[m]$ then a sequence $\sigma = (i_1, \dots, i_k)$ is a *missing face* of K if $\sigma \notin K$. It is a *minimal missing face* of K if no proper subsequence of σ is a missing face of K .

Theorem 1.1. *Let K be a simplicial complex on the vertex set $[m]$ and let $(\underline{X}, \underline{A})$ be any sequence of pairs (D^{n_i}, S^{n_i-1}) with $n_i \geq 2$ for $1 \leq i \leq m$. Then:*

- (a) $(\underline{X}, \underline{A})^K$ is elliptic if and only if the minimal missing faces of K are mutually disjoint;
- (b) Moore's conjecture holds for $(\underline{X}, \underline{A})^K$.

In particular, Theorem 1.1 includes generalized moment-angle complexes $\mathcal{Z}_K(D^n, S^{n-1})$ for $n \geq 2$ as a special case. Part (a) of Theorem 1.1 was proved by [1] in the special case of the moment-angle complex \mathcal{Z}_K , although part (b) was not. The restriction to $n \geq 2$ is made to ensure that certain retractions constructed in Theorem 4.2 involve wedges of simply-connected spheres which are hyperbolic, rather than wedge of circles which are Eilenberg-MacLane spaces.

We also give a general criterion for when a polyhedral product is elliptic or hyperbolic. This generalizes and reformulates in more combinatorial terms results obtained by Félix and Tanré [8].

Theorem 1.2. *Let K be a simplicial complex on the vertex set $[m]$ and let $(\underline{X}, \underline{A})$ be any sequence of pairs. For $1 \leq i \leq m$, let Y_i be the homotopy fibre of the inclusion $A_i \rightarrow X_i$ and suppose that each Y_i is rationally nontrivial. Then the polyhedral product $(\underline{X}, \underline{A})^K$ is elliptic if and only if three conditions hold:*

- (i) *each X_i is elliptic;*
- (ii) *all the minimal missing faces of K are mutually disjoint;*
- (iii) *if v is a vertex of a minimal missing face of K then Y_v is rationally homotopy equivalent to a sphere.*

2. POLYHEDRAL PRODUCT INGREDIENTS

This section contains the properties of polyhedral products that will be needed. We begin with two results of Denham and Suciu [6], stated as Propositions 2.1 and 2.2.

Proposition 2.1. *Let K be a simplicial complex on the vertex set $[m]$. Suppose that for $1 \leq i \leq m$ there are maps of pairs $p_i: (E_i, E'_i) \rightarrow (B_i, B_i)$ such that the restrictions $E_i \rightarrow B_i$ and $E'_i \rightarrow B_i$ are fibrations with fibres F_i and F'_i respectively. Then there is a fibration*

$$(\underline{E}, \underline{F}')^K \rightarrow (\underline{E}, \underline{E}')^K \rightarrow \prod_{i=1}^m B_i.$$

□

Let K be a simplicial complex on the vertex set $[m]$. If $I \subseteq [m]$ then the *full subcomplex* K_I of K is defined as the simplicial complex

$$K_I = \bigcup \{ \sigma \in K \mid \text{the vertex set of } \sigma \text{ is in } I \}.$$

The definition of K_I implies that the inclusion $K_I \rightarrow K$ is a map of simplicial complexes. This induces a map of polyhedral products $(\underline{X}, \underline{A})^{K_I} \rightarrow (\underline{X}, \underline{A})^K$. There is no retraction of K_I off K as simplicial complexes, however, in [6] it was shown that there is nevertheless a retraction of $(\underline{X}, \underline{A})^{K_I}$ off $(\underline{X}, \underline{A})^K$.

Proposition 2.2. *Let K be a simplicial complex on the vertex set $[m]$ and let $(\underline{X}, \underline{A})$ be any sequence of pointed, path-connected CW-pairs. Let $I \subseteq [m]$. Then the inclusion $(\underline{X}, \underline{A})^{K_I} \rightarrow (\underline{X}, \underline{A})^K$ has a left inverse.*

□

Proposition 2.1 will be applied to prove the existence of a certain fibration in Theorem 2.3, and Proposition 2.2 will be used to show that it splits after looping. To set up, a procedure is described for turning a map of pairs $(X, A) \rightarrow (X, X)$ into a map of pairs of fibrations, up to homotopy. This is analogous to how a continuous map of spaces is turned into a fibration, up to homotopy.

Let (X, A) be a pair of pointed, path-connected spaces and let $I = [0, 1]$ be the unit interval. Define the spaces P^j and P^{id} by the topological (strict) pullbacks

$$\begin{array}{ccc} P^j & \longrightarrow & X^I \\ \downarrow \pi_2 & & \downarrow ev_0 \\ A & \xrightarrow{j} & X \end{array} \qquad \begin{array}{ccc} P^{id} & \longrightarrow & X^I \\ \downarrow \pi_2 & & \downarrow ev_0 \\ X & \xrightarrow{id} & X \end{array}$$

where j is the inclusion, id is the identity map, π_2 is the projection, X^I is the space of continuous paths (not necessarily pointed) from I to X , and $ev_0: X^I \rightarrow X$ is the map that evaluates a path in X at 0. Observe that

$$P^j = \{(\omega, a) \in X^I \times A \mid \omega(0) = a\} \quad P^{id} = \{(\omega, x) \in X^I \times X \mid \omega(0) = x\}.$$

Observe that the inclusion $X^I \times A \xrightarrow{1 \times j} X^I \times X$ implies that P^j is a subspace of P^{id} . Further, since ev_0 is a homotopy equivalence, the pullback implies that the maps labelled π_2 are as well, and these can be chosen to be compatible, so we obtain a homotopy equivalence of pairs $(P^{id}, P^j) \rightarrow (X, A)$. Moreover, the composites $P^j \rightarrow X^I \xrightarrow{ev_1} X$ and $P^{id} \rightarrow X^I \xrightarrow{ev_1} X$ are fibrations, where ev_1 evaluates a path at 1. Thus the map of pairs $(P^{id}, P^j) \rightarrow (X, X)$ is such that the restrictions to $P^{id} \rightarrow X$ and $P^j \rightarrow X$ are fibrations.

Next, consider the fibres in each case. They are

$$F^j = \{(\omega, a) \in X^I \times A \mid \omega(0) = a \text{ and } \omega(1) = *\} \quad F^{id} = \{(\omega, x) \in X^I \times X \mid \omega(0) = x \text{ and } \omega(1) = *\}$$

respectively. Recall that the *path space* of X is $PX = \{\omega \in X^I \mid \omega(1) = *\}$. This condition on paths appears in each of F^j and F^{id} , implying that we in fact have

$$(1) \quad F^j = \{(\omega, a) \in PX \times A \mid \omega(0) = a\} \quad F^{id} = \{(\omega, x) \in PX \times X \mid \omega(0) = x\}.$$

Observe that the inclusion $PX \times A \xrightarrow{1 \times j} PX \times X$ implies that F^j is a subspace of F^{id} , so there is a pair (F^{id}, F^j) . Observe also that as F^{id} is homotopy equivalent to the homotopy fibre of the identity map on X , F^{id} is contractible.

We wish to express the pair (F^{id}, F^j) in terms of the cone on F^j and F^j , which is more standard in toric topology. For a pointed space Y , the *reduced cone* on Y is the quotient space $CY = (Y \times I) / \sim$, where $(y, 1) \sim (y', 1)$ for all $y, y' \in Y$ and $(*, t) \simeq (*, 0)$ for all $t \in I$. Observe that Y may be regarded as a subspace of CY by sending $y \in Y$ to $(y, 0) \in CY$. In our case, CF^j is contractible so there is a homotopy equivalence $F^{id} \simeq CF^j$. But we wish to choose this homotopy equivalence more carefully so that we obtain a homotopy equivalence of pairs $(F^{id}, F^j) \rightarrow (CF^j, F^j)$. To do this, we show that both F^{id} and CF^j are subspaces of a larger contractible space Q . Observe that by (1), F^{id} is a subspace of $PX \times X$, which in turn is a subspace of $CPX \times CX$. On the other hand, by (1), F^j is a subspace of $PX \times A$, so CF^j is a subspace of $C(PX \times A)$, which in turn is a subspace of $CPX \times CA$. Since A is a subspace of X , $CPX \times CA$ is a subspace of $CPX \times CX$. So take $Q = CPX \times CX$.

Doing so, there are maps of pairs $(F^{id}, F^j) \rightarrow (Q, F^j)$ and $(CF^j, F^j) \rightarrow (Q, F^j)$ which induce maps of long exact sequences in reduced homology

$$\begin{array}{ccccccc}
\cdots & \longrightarrow & \tilde{H}_*(F^j) & \longrightarrow & \tilde{H}_*(F^{id}) & \longrightarrow & \tilde{H}_*((F^{id}, F^j)) \longrightarrow \cdots \\
& & \downarrow = & & \downarrow & & \downarrow \\
\cdots & \longrightarrow & \tilde{H}_*(F^j) & \longrightarrow & \tilde{H}_*(Q) & \longrightarrow & \tilde{H}_*((Q, F^j)) \longrightarrow \cdots \\
& & \downarrow = & & \downarrow & & \downarrow \\
\cdots & \longrightarrow & \tilde{H}_*(F^j) & \longrightarrow & \tilde{H}_*(CF^j) & \longrightarrow & \tilde{H}_*((CF^j, F^j)) \longrightarrow \cdots \\
& & \downarrow = & & \downarrow & & \downarrow \\
\cdots & \longrightarrow & \tilde{H}_*(F^j) & \longrightarrow & \tilde{H}_*(Q) & \longrightarrow & \tilde{H}_*((Q, F^j)) \longrightarrow \cdots
\end{array}$$

Since F^{id} , CF^j and Q are all contractible, the middle maps in each diagram are isomorphisms. The five-lemma then implies that the right-hand maps are also isomorphisms. Hence we obtain homotopy equivalences of pairs $(F^{id}, F^j) \rightarrow (Q, F^j)$ and $(CF^j, F^j) \rightarrow (Q, F^j)$, which implies that there is a homotopy equivalence of pairs $(F^{id}, F^j) \rightarrow (CF^j, F^j)$, as desired.

Finally, we are ready to state and prove the following theorem.

Theorem 2.3. *Let K be a simplicial complex on the vertex set $[m]$ and let $(\underline{X}, \underline{A})$ be any sequence of pointed, path-connected CW-pairs. Then there is a homotopy fibration*

$$(\underline{CY}, \underline{Y})^K \rightarrow (\underline{X}, \underline{A})^K \rightarrow \prod_{i=1}^m X_i$$

where, for $1 \leq i \leq m$, Y_i is the homotopy fibre of the inclusion $A_i \rightarrow X_i$, and a homotopy equivalence

$$\Omega(\underline{X}, \underline{A})^K \simeq \left(\prod_{i=1}^m \Omega X_i \right) \times \Omega(\underline{CY}, \underline{Y})^K.$$

Proof. For $1 \leq i \leq m$, there are maps of pairs $(X_i, A_i) \rightarrow (X_i, X_i)$. Use the procedure described above to replace these pairs, up to homotopy, by the map of pairs $(P_i^{id}, P_i^j) \rightarrow (X_i, X_i)$ with the property that the restrictions $P_i^{id} \rightarrow X_i$ and $P_i^j \rightarrow X_i$ are fibrations with fibres F_i^{id} and F_i^j respectively. By Proposition 2.1 there is a fibration

$$(\underline{F}^{id}, \underline{F}^j)^K \rightarrow (\underline{P}^{id}, \underline{P}^j) \rightarrow (X_i, X_i)^K.$$

By definition of the polyhedral product, $(X_i, X_i)^K = \prod_{i=1}^m X_i$. For each $1 \leq i \leq m$, there is a homotopy equivalence of pairs $(P_i^{id}, P_i^j) \simeq (X_i, A_i)$, and the construction preceding the statement of the lemma shows that there is also a homotopy equivalence of pairs $(F_i^{id}, F_i^j) \simeq (CF_i^j, F_i^j)$. Let $Y_i = F_i^j$. Then there is a homotopy fibration

$$(2) \quad (\underline{CY}, \underline{Y})^K \rightarrow (\underline{X}, \underline{A})^K \rightarrow \prod_{i=1}^m X_i$$

where Y_i is the homotopy fibre of the inclusion $A_i \rightarrow X_i$.

Next, we wish to show that the homotopy fibration (2) splits after looping. For $1 \leq i \leq m$, let $I_i = \{i\}$. Observe that the full subcomplex K_{I_i} of K is just the vertex $\{i\}$. By the definition of the polyhedral product, $(\underline{X}, \underline{A})^{K_{I_i}} = X_i$. Proposition 2.2 therefore implies that X_i retracts off $(\underline{X}, \underline{A})^K$. Explicitly, the composite $X_i = (\underline{X}, \underline{A})^{K_{I_i}} \rightarrow (\underline{X}, \underline{A})^K \rightarrow \prod_{i=1}^m X_i \xrightarrow{\text{proj}} X_i$ is the identity map. After looping, the loop maps $\Omega X_i \rightarrow \Omega(\underline{X}, \underline{A})^K$ may be multiplied together to obtain a map $\prod_{i=1}^m \Omega X_i \rightarrow \Omega(\underline{X}, \underline{A})^K$ which is a right homotopy inverse of the map $\Omega(\underline{X}, \underline{A})^K \rightarrow \prod_{i=1}^m \Omega X_i$. Hence, if μ is the loop multiplication on $\Omega(\underline{X}, \underline{A})^K$, then the composite

$$\left(\prod_{i=1}^m \Omega X_i\right) \times \Omega(\underline{CY}, \underline{Y})^K \rightarrow \Omega(\underline{X}, \underline{A})^K \times \Omega(\underline{X}, \underline{A})^K \xrightarrow{\mu} \Omega(\underline{X}, \underline{A})^K$$

is a homotopy equivalence. \square

Theorem 2.3 implies that homotopy group information about $(\underline{X}, \underline{A})^K$ is determined by that of the ingredient spaces X_i and $(\underline{CY}, \underline{Y})^K$. This is useful because the spaces $(\underline{CY}, \underline{Y})^K$ are much better understood than the spaces $(\underline{X}, \underline{A})^K$.

The following proposition, proved in [10], relates pushouts of simplicial complexes to pushouts of polyhedral products.

Proposition 2.4. *Let K be a simplicial complex on the vertex set $[m]$. Suppose that there is a pushout of simplicial complexes*

$$\begin{array}{ccc} L & \longrightarrow & K_2 \\ \downarrow & & \downarrow \\ K_1 & \longrightarrow & K \end{array}$$

Let L° , K_1° and K_2° be L , K_1 and K_2 regarded as simplicial complexes on the same vertex set as K . Then there is a pushout of polyhedral products

$$\begin{array}{ccc} (\underline{X}, \underline{A})^{L^\circ} & \longrightarrow & (\underline{X}, \underline{A})^{K_2^\circ} \\ \downarrow & & \downarrow \\ (\underline{X}, \underline{A})^{K_1^\circ} & \longrightarrow & (\underline{X}, \underline{A})^K. \end{array}$$

\square

Finally, we give two examples where the homotopy type of $(\underline{CY}, \underline{Y})^K$ is explicitly identified. Part (a) in Lemma 2.5 is immediate from the definition of the polyhedral product, while part (b) was proved by Porter [17] when each Y_i is a loop space and more generally in [10].

Lemma 2.5. *Let Y_1, \dots, Y_m be path-connected spaces. Then the following hold:*

- (a) $(\underline{CY}, \underline{Y})^{\Delta^{m-1}} = \prod_{i=1}^m CY_i$;
- (b) $(\underline{CY}, \underline{Y})^{\partial \Delta^{m-1}} \simeq \Sigma^{m-1} Y_1 \wedge \dots \wedge Y_m$.

\square

3. COMBINATORIAL INGREDIENTS

This section records the combinatorial information that will be needed. Let K be a simplicial complex on the index set $[m]$. For a vertex $v \in K$, the *star*, *restriction* (or *deletion*) and *link* of v are the subcomplexes

$$\begin{aligned} \text{star}_K(v) &= \{\tau \in K \mid \{v\} \cup \tau \in K\}; \\ K \setminus v &= \{\tau \in K \mid \{v\} \cap \tau = \emptyset\}; \\ \text{link}_K(v) &= \text{star}_K(v) \cap K \setminus v. \end{aligned}$$

The *join* of two simplicial complexes K_1, K_2 on disjoint index sets is the simplicial complex

$$K_1 * K_2 = \{\sigma_1 \cup \sigma_2 \mid \sigma_i \in K_i\}.$$

From the definitions, it follows that $\text{star}_K(v)$ is a join,

$$\text{star}_K(v) = \{v\} * \text{link}_K(v),$$

and there is a pushout

$$\begin{array}{ccc} \text{link}_K(v) & \longrightarrow & \text{star}_K(v) \\ \downarrow & & \downarrow \\ K \setminus v & \longrightarrow & K. \end{array}$$

A *face* of K is a simplex of K . Let Δ^{m-1} be the standard m -simplex on the vertex set $[m]$ and note that K is a subcomplex of Δ^{m-1} . Recall from the Introduction that a face $\sigma \in \Delta^{m-1}$ is a *missing face* of K if $\sigma \notin K$. It is a *minimal missing face* if any proper face of σ is a face of K . Denote the set of minimal missing faces of K by $MMF(K)$. For a simplex σ , let $\partial\sigma$ be its boundary. Observe that $\sigma \in MMF(K)$ if and only if $\sigma \notin K$ but $\partial\sigma \subseteq K$.

There is a special case which will play a crucial role in what follows. Let \overline{K} be a simplicial complex on the vertex set $[m]$ with the property that it has precisely two distinct minimal missing faces and these have non-empty intersection. That is, suppose that $MMF(\overline{K}) = \{\sigma_1, \sigma_2\}$ where σ_1 and σ_2 have vertex sets I and J respectively, satisfying $I \neq J$, $I \cup J = [m]$ and $I \cap J \neq \emptyset$. Let w be a vertex in both I and J .

Consider the star-link-restriction pushout of \overline{K} with respect to the vertex w :

$$(3) \quad \begin{array}{ccc} \text{link}_{\overline{K}}(w) & \longrightarrow & \text{star}_{\overline{K}}(w) \\ \downarrow & & \downarrow \\ \overline{K} \setminus w & \longrightarrow & \overline{K}. \end{array}$$

Let $\overline{\sigma}_1$ and $\overline{\sigma}_2$ be the proper faces of σ_1 and σ_2 on the vertex sets $\overline{I} = I \setminus \{w\}$ and $\overline{J} = J \setminus \{w\}$ respectively.

Lemma 3.1. *We have $\overline{\sigma}_1, \overline{\sigma}_2 \in MMF(\text{star}_{\overline{K}}(w))$.*

Proof. Consider $\bar{\sigma}_1$, the argument for $\bar{\sigma}_2$ being similar. First we show that $\bar{\sigma}_1$ is a missing face of $\text{star}_{\bar{K}}(w)$. For if $\bar{\sigma}_1 \in \text{star}_{\bar{K}}(w)$ then, as w is not a vertex of $\bar{\sigma}_1$, we also have $\bar{\sigma}_1 \in \bar{K} \setminus w$, implying that $\bar{\sigma}_1 \in \text{link}_{\bar{K}}(w) = \text{star}_{\bar{K}}(w) \cap \bar{K} \setminus w$. This in turn implies that $\bar{\sigma}_1 * \{w\} \in \text{star}_{\bar{K}}(w)$. But $\bar{\sigma}_1 * \{w\} = \sigma_1$, so $\sigma_1 \in \text{star}_{\bar{K}}(w)$. Therefore, by (3), $\sigma_1 \in \bar{K}$, contradicting the fact that σ_1 is a missing face of \bar{K} .

Next, we show that that $\bar{\sigma}_1$ is a minimal missing face of $\text{star}_{\bar{K}}(w)$. If not, then some proper face τ of $\bar{\sigma}_1$ is also a missing face of $\text{star}_{\bar{K}}(w)$. As w is not a vertex of $\bar{\sigma}_1$, it is not a vertex of τ either. Therefore $\tau * \{w\}$ is a missing face of $\text{star}_{\bar{K}}(w)$. The presence of the vertex w in $\tau * \{w\}$ implies that it is also not a face of $\bar{K} \setminus w$. On the other hand, by (3), \bar{K} is the union of $\text{star}_{\bar{K}}(w)$ and $\bar{K} \setminus w$, so a face that is missing from both $\text{star}_{\bar{K}}(w)$ and $\bar{K} \setminus w$ must also be missing from \bar{K} . Therefore $\tau * \{w\}$ is a missing face of \bar{K} . But as τ is a proper face of $\bar{\sigma}_1$, $\tau * \{w\}$ is a proper face of $\bar{\sigma}_1 * \{w\} = \sigma_1$, contradicting the fact that σ_1 is a minimal missing face of \bar{K} . \square

Corollary 3.2. *We have $\partial\bar{\sigma}_1, \partial\bar{\sigma}_2 \subseteq \text{link}_{\bar{K}}(w)$ and $\bar{\sigma}_1, \bar{\sigma}_2 \notin \text{link}_{\bar{K}}(w)$.*

Proof. Recall that a face σ of a simplicial complex K is a minimal missing face if and only if $\sigma \notin K$ but $\partial\sigma \subseteq K$. So by Lemma 3.1, $\partial\bar{\sigma}_1, \partial\bar{\sigma}_2 \subseteq \text{star}_{\bar{K}}(w)$. By definition, neither $\bar{\sigma}_1$ nor $\bar{\sigma}_2$ have w in their vertex sets, so neither do their boundaries. Therefore $\partial\bar{\sigma}_1, \partial\bar{\sigma}_2 \subseteq \bar{K} \setminus w$. Therefore, as $\text{link}_{\bar{K}}(w) = \text{star}_{\bar{K}}(w) \cap \bar{K} \setminus w$, we have $\partial\bar{\sigma}_1, \partial\bar{\sigma}_2 \subseteq \text{link}_{\bar{K}}(w)$.

Also, as $\text{link}_{\bar{K}}(w) = \text{star}_{\bar{K}}(w) \cap \bar{K} \setminus w$, it cannot be that $\bar{\sigma}_1, \bar{\sigma}_2$ are in $\text{link}_{\bar{K}}(w)$ as that would imply they are also in $\text{star}_{\bar{K}}(w)$, contradicting Lemma 3.1. \square

One further observation we need regarding \bar{K} is the following. Regarding w as the m^{th} -vertex of \bar{K} , observe that $\bar{K} \setminus w$ is a simplicial complex on the vertex set $[m-1]$.

Lemma 3.3. *There is an isomorphism of simplicial complexes $\bar{K} \setminus w \cong \Delta^{m-2}$.*

Proof. It is equivalent to show that $\bar{K} \setminus w$ has no missing faces. Suppose that $\sigma \in \Delta^{m-2}$ is a missing face of $\bar{K} \setminus w$. Then as $\bar{K} \setminus w$ is the restriction of \bar{K} to the vertex set $[m-1]$, σ is also a missing face of \bar{K} . On the other hand, as $\text{MMF}(\bar{K}) = \{\sigma_1, \sigma_2\}$, any missing face of \bar{K} must have either σ_1 or σ_2 as a subface. Thus σ must have either σ_1 or σ_2 as a subface. But this cannot happen since w is not in the vertex set of σ but it is in the vertex sets of both σ_1 and σ_2 . \square

4. MOORE'S CONJECTURE

In this section we prove Theorems 1.1 and 1.2 as consequences of Theorem 4.2.

Proposition 4.1. *Let K be a simplicial complex on the vertex set $[m]$ and let X_1, \dots, X_m be any sequence of pointed, path-connected CW-pairs. Suppose that $\sigma_1, \sigma_2 \in \text{MMF}(K)$ and let I and J be the vertex sets of σ_1 and σ_2 respectively. If $I \neq J$, $I \cup J = [m]$ and $I \cap J \neq \emptyset$, then $(\underline{CX}, \underline{X})^{\partial\sigma_1} \vee (\underline{CX}, \underline{X})^{\partial\sigma_2}$ is a retract of $(\underline{CX}, \underline{X})^K$.*

Proof. A new simplicial complex is introduced that will act as an intermediary. In general, a simplicial complex may be characterized by listing its minimal missing faces. Let \overline{K} be the simplicial complex on the vertex set $[m]$ that is characterized by the condition that $MMF(\overline{K}) = \{\sigma_1, \sigma_2\}$. Intuitively, \overline{K} is obtained from K by filling in all missing faces that do not have either σ_1 or σ_2 as a subface. Rigorously, there is a map of simplicial complexes $K \rightarrow \overline{K}$ that induces a map of polyhedral products $(\underline{CX}, \underline{X})^K \rightarrow (\underline{CX}, \underline{X})^{\overline{K}}$. Since σ_1, σ_2 are minimal missing faces of K , we have $\sigma_1, \sigma_2 \notin K$ but $\partial\sigma_1, \partial\sigma_2 \subseteq K$. The inclusion $\partial\sigma_1 \rightarrow K$ is a map of simplicial complexes and it induces a map of polyhedral products $(\underline{CX}, \underline{X})^{\partial\sigma_1} \rightarrow (\underline{CX}, \underline{X})^K$. There is a similar map with respect to $\partial\sigma_2$. We will show that the composite $(\underline{CX}, \underline{X})^{\partial\sigma_1} \vee (\underline{CX}, \underline{X})^{\partial\sigma_2} \rightarrow (\underline{CX}, \underline{X})^K \rightarrow (\underline{CX}, \underline{X})^{\overline{K}}$ has a left homotopy inverse. Note that this composite of polyhedral products is the same as the one induced by the inclusions $\partial\sigma_1 \rightarrow \overline{K}$ and $\partial\sigma_2 \rightarrow \overline{K}$, so it suffices to show that the map $(\underline{CX}, \underline{X})^{\partial\sigma_1} \vee (\underline{CX}, \underline{X})^{\partial\sigma_2} \rightarrow (\underline{CX}, \underline{X})^{\overline{K}}$ has a left homotopy inverse.

The conditions on the vertex sets I and J imply that \overline{K} has the same form as in Section 3. Relabelling the spaces X_1, \dots, X_m if necessary, we may suppose that the intersection vertex w corresponds to the m^{th} -coordinate space X_m . By Proposition 2.4, the pushout of simplicial complexes in (3) implies that there is a pushout of polyhedral products

$$(4) \quad \begin{array}{ccc} (\underline{CX}, \underline{X})^{\text{link}_{\overline{K}}(w)^\circ} & \xrightarrow{g^\circ} & (\underline{CX}, \underline{X})^{\text{star}_{\overline{K}}(w)^\circ} \\ \downarrow f^\circ & & \downarrow \\ (\underline{CX}, \underline{X})^{\overline{K} \setminus w^\circ} & \longrightarrow & (\underline{CX}, \underline{X})^{\overline{K}}. \end{array}$$

where $\text{link}_{\overline{K}}(w)^\circ$, $\text{star}_{\overline{K}}(w)^\circ$ and $\overline{K} \setminus w^\circ$ are $\text{link}_{\overline{K}}(w)$, $\text{star}_{\overline{K}}(w)$ and $\overline{K} \setminus w$ regarded as having vertex set $[m]$, and the maps f° and g° are induced by the inclusions $\text{link}_{\overline{K}}(w)^\circ \rightarrow \overline{K} \setminus w^\circ$ and $\text{link}_{\overline{K}}(w)^\circ \rightarrow \text{star}_{\overline{K}}(w)^\circ$ respectively. The vertex sets of $\text{link}_{\overline{K}}(w)$ and $\overline{K} \setminus w$ are both $[m-1]$, so by the definition of the polyhedral product,

$$(\underline{CX}, \underline{X})^{\text{link}_{\overline{K}}(w)^\circ} = (\underline{CX}, \underline{X})^{\text{link}_{\overline{K}}(w)} \times X_m \quad (\underline{CX}, \underline{X})^{\overline{K} \setminus w^\circ} = (\underline{CX}, \underline{X})^{\overline{K} \setminus w} \times X_m$$

and $f^\circ = f \times 1$ where f is induced by the inclusion $\text{link}_{\overline{K}}(w) \rightarrow \overline{K} \setminus w$ and 1 is the identity map on X_m . On the other hand, the vertex set of $\text{star}_{\overline{K}}(w)$ is $[m]$ so $\text{star}_{\overline{K}}(w)^\circ = \text{star}_{\overline{K}}(w)$. Since $\text{star}_{\overline{K}}(w) = \text{link}_{\overline{K}}(w) * \{w\}$, the definition of the polyhedral product implies that

$$(\underline{CX}, \underline{X})^{\text{star}_{\overline{K}}(w)^\circ} = (\underline{CX}, \underline{X})^{\text{link}_{\overline{K}}(w)} \times CX_m$$

and $g^\circ = 1 \times i_m$ where 1 is the identity map on $(\underline{CX}, \underline{X})^{\text{link}_{\overline{K}}(w)}$ and $i_m: X_m \rightarrow CX_m$ is the inclusion of the base of the cone. Putting all this together, the pushout (4) becomes the pushout

$$(5) \quad \begin{array}{ccc} (\underline{CX}, \underline{X})^{\text{link}_{\overline{K}}(w)} \times X_m & \xrightarrow{1 \times i_m} & (\underline{CX}, \underline{X})^{\text{link}_{\overline{K}}(w)} \times CX_m \\ \downarrow f \circ 1 & & \downarrow \\ (\underline{CX}, \underline{X})^{\overline{K} \setminus w} \times X_m & \longrightarrow & (\underline{CX}, \underline{X})^{\overline{K}}. \end{array}$$

By Lemma 3.3, $\overline{K} \setminus w \cong \Delta^{m-2}$, so by Lemma 2.5 (a), $(\underline{CX}, \underline{X})^{\overline{K} \setminus w} = \prod_{i=1}^{m-1} CX_i$. Therefore, in (5), both $(\underline{CX}, \underline{X})^{\overline{K} \setminus w}$ and CX_m are contractible, implying that (5) is equivalent, up to homotopy, to the homotopy pushout

$$(6) \quad \begin{array}{ccc} (\underline{CX}, \underline{X})^{\text{link}_{\overline{K}}(w)} \times X_m & \xrightarrow{\pi_1} & (\underline{CX}, \underline{X})^{\text{link}_{\overline{K}}(w)} \\ \downarrow \pi_2 & & \downarrow \\ X_m & \longrightarrow & (\underline{CX}, \underline{X})^{\overline{K}} \end{array}$$

where π_1 and π_2 are the projections onto the first and second factors respectively. It is well known that the pushout of the projections $A \times B \rightarrow A$ and $A \times B \rightarrow B$ is homotopy equivalent to the join of A and B , which in turn is homotopy equivalent to $\Sigma A \wedge B$. So (6) implies that there is a homotopy equivalence

$$(7) \quad (\underline{CX}, \underline{X})^{\overline{K}} \simeq \Sigma(\underline{CX}, \underline{X})^{\text{link}_{\overline{K}}(w)} \wedge X_m.$$

Now consider the minimal missing faces σ_1 and σ_2 of \overline{K} . As in Section 3, let $\overline{\sigma}_1, \overline{\sigma}_2$ be the restrictions of σ_1, σ_2 respectively to the vertex sets $\overline{I} = I \setminus \{w\}, \overline{J} = J \setminus \{w\}$. Note that as $I \neq J$ we also have $\overline{I} \neq \overline{J}$. By Corollary 3.2, $\overline{\sigma}_1, \overline{\sigma}_2 \notin \text{link}_{\overline{K}}(w)$ but $\partial \overline{\sigma}_1, \partial \overline{\sigma}_2 \subseteq \text{link}_{\overline{K}}(w)$. Therefore, the full subcomplex of $\text{link}_{\overline{K}}(w)$ on \overline{I} is $\partial \overline{\sigma}_1$, and the full subcomplex of $\text{link}_{\overline{K}}(w)$ on \overline{J} is $\partial \overline{\sigma}_2$. By Proposition 2.2, this implies that $(\underline{CX}, \underline{X})^{\partial \overline{\sigma}_1}$ and $(\underline{CX}, \underline{X})^{\partial \overline{\sigma}_2}$ are retracts of $(\underline{CX}, \underline{X})^{\text{link}_{\overline{K}}(w)}$. By [1, Theorem 2.21], the fact that $\partial \overline{\sigma}_1$ and $\partial \overline{\sigma}_2$ are full subcomplexes of $\text{link}_{\overline{K}}(w)$ on different index sets implies that $\Sigma(\underline{CX}, \underline{X})^{\partial \overline{\sigma}_1} \vee \Sigma(\underline{CX}, \underline{X})^{\partial \overline{\sigma}_2}$ is a retract of $\Sigma(\underline{CX}, \underline{X})^{\text{link}_{\overline{K}}(w)}$. Thus (7) implies that $(\Sigma(\underline{CX}, \underline{X})^{\partial \overline{\sigma}_1} \wedge X_m) \vee (\Sigma(\underline{CX}, \underline{X})^{\partial \overline{\sigma}_2} \wedge X_m)$ is a retract of $(\underline{CX}, \underline{X})^{\overline{K}}$.

We wish to choose the retraction more carefully. Restrict \overline{K} to the full subcomplex on the vertex set I . Then $MMF(\overline{K}_I) = \{\sigma_1\}$, so $\overline{K}_I = \partial \sigma_1$. Therefore the star-link-restriction pushout for \overline{K}_I with respect to the vertex w becomes

$$\begin{array}{ccc} \partial \overline{\sigma}_1 & \longrightarrow & \partial \overline{\sigma}_1 * \{w\} \\ \downarrow & & \downarrow \\ \partial \sigma_1 \setminus w & \longrightarrow & \partial \sigma_1. \end{array}$$

Note that $\partial \sigma_1 \setminus w$ is the simplex Δ^{k-1} on the vertex set $\{i_1, \dots, i_k\}$. Now arguing as for (4) – (6) and equation (7), we obtain in place of (7) a homotopy equivalence $(\underline{CX}, \underline{X})^{\overline{K}_I} = (\underline{CX}, \underline{X})^{\partial \sigma_1} \simeq$

$\Sigma(\underline{CX}, \underline{X})^{\partial\sigma_1} \wedge X_m$. Thus we may choose the map $\Sigma(\underline{CX}, \underline{X})^{\partial\sigma_1} \wedge X_m \rightarrow (\underline{CX}, \underline{X})^{\overline{K}}$ as the composite $\Sigma(\underline{CX}, \underline{X})^{\partial\sigma_1} \wedge X_m \xrightarrow{\simeq} (\underline{CX}, \underline{X})^{\partial\sigma_1} \rightarrow (\underline{CX}, \underline{X})^{\overline{K}}$. Doing the same for $\partial\sigma_2$ we obtain a composite $(\Sigma(\underline{CX}, \underline{X})^{\partial\sigma_1} \wedge X_m) \vee (\Sigma(\underline{CX}, \underline{X})^{\partial\sigma_2} \wedge X_m) \xrightarrow{\simeq} (\underline{CX}, \underline{X})^{\partial\sigma_1} \vee (\underline{CX}, \underline{X})^{\partial\sigma_2} \rightarrow (\underline{CX}, \underline{X})^{\overline{K}}$, and it is this composite that has a left homotopy inverse. In particular, we have produced a left homotopy inverse for the map $(\underline{CX}, \underline{X})^{\partial\sigma_1} \vee (\underline{CX}, \underline{X})^{\partial\sigma_2} \rightarrow (\underline{CX}, \underline{X})^{\overline{K}}$ induced by the inclusions $\partial\sigma_1 \rightarrow \overline{K}$ and $\partial\sigma_2 \rightarrow \overline{K}$, as required. \square

Recall that for $1 \leq i \leq m$, Y_i is the homotopy fibre of the inclusion $A_i \rightarrow X_i$.

Theorem 4.2. *Let K be a simplicial complex on the vertex set $[m]$ and let $(\underline{X}, \underline{A})$ be any sequence of pointed, path-connected CW-pairs. The following hold:*

- (a) *if $MMF(K) = \{\sigma_1, \dots, \sigma_n\}$ and these minimal missing faces are mutually disjoint, then there is a homotopy equivalence*

$$\Omega(\underline{X}, \underline{A})^K \simeq \left(\prod_{i=1}^m \Omega X_i \right) \times \left(\prod_{j=1}^n \Omega(\underline{CY}, \underline{Y})^{\partial\sigma_j} \right);$$

- (b) *if σ_1 and σ_2 are minimal missing faces of K with nontrivial intersection then $\Omega((\underline{CY}, \underline{Y})^{\partial\sigma_1} \vee (\underline{CY}, \underline{Y})^{\partial\sigma_2})$ retracts off $\Omega(\underline{X}, \underline{A})^K$.*

Proof. By Theorem 2.3, there is a homotopy equivalence

$$(8) \quad \Omega(\underline{X}, \underline{A})^K \simeq \left(\prod_{i=1}^m \Omega X_i \right) \times \Omega(\underline{CY}, \underline{Y})^K$$

where, for $1 \leq i \leq m$, Y_i is the homotopy fibre of the inclusion $A_i \rightarrow X_i$.

If all of the minimal missing faces of K are mutually disjoint then there is a simplicial isomorphism $K \cong K_0 * K_1 * \dots * K_n$ where K_0 is a product of simplices and, for $1 \leq j \leq n$, $K_j = \partial\sigma_j$ (a proof of this may be found in [2], although it may be more commonly known). In general, the definition of a polyhedral product implies that there is a homeomorphism $(\underline{X}, \underline{A})^{L*M} \cong (\underline{X}, \underline{A})^L \times (\underline{X}, \underline{A})^M$. In our case, as K_0 is a simplex, Lemma 2.5 (a) implies that $(\underline{CY}, \underline{Y})^{K_0}$ is a product of cones and so is contractible. Thus

$$(\underline{CY}, \underline{Y})^K \simeq (\underline{CY}, \underline{Y})^{K_1} \times \dots \times (\underline{CY}, \underline{Y})^{K_n} = (\underline{CY}, \underline{Y})^{\partial\sigma_1} \times \dots \times (\underline{CY}, \underline{Y})^{\partial\sigma_n}.$$

Combining this with (8), the homotopy decomposition in part (a) follows.

Next, suppose that σ_1 and σ_2 are minimal missing faces of K that intersect nontrivially. Let I and J be the vertex sets of σ_1 and σ_2 respectively. Let $K_{I \cup J}$ be the full subcomplex of K on the index set $I \cup J$. By Proposition 2.2, $(\underline{CY}, \underline{Y})^{K_{I \cup J}}$ is a retract of $(\underline{CY}, \underline{Y})^K$. Further, Proposition 4.1 implies that $(\underline{CY}, \underline{Y})^{\partial\sigma_1} \vee (\underline{CY}, \underline{Y})^{\partial\sigma_2}$ is a retract of $(\underline{CY}, \underline{Y})^{K_{I \cup J}}$. Hence $(\underline{CY}, \underline{Y})^{\partial\sigma_1} \vee (\underline{CY}, \underline{Y})^{\partial\sigma_2}$ is a retract of $(\underline{CY}, \underline{Y})^K$. Combining this with (8), the assertion in part (b) follows. \square

We now turn to Moore's Conjecture and the distinguishing of elliptic and hyperbolic spaces. For Theorem 1.1, we assume that each pair (X_i, A_i) is (D^{n_i}, S^{n_i-1}) for $n_i \geq 2$. Note that the

homotopy fibre Y_i of the inclusion $S^{n_i-1} \rightarrow D^{n_i}$ is also S^{n_i-1} , so the pair (CY_i, Y_i) in Theorem 4.2 is also homotopy equivalent to (D^{n_i}, S^{n_i-1}) . Note that as each X_i is D^{n_i} , the term $\prod_{i=1}^m \Omega X_i$ in Theorem 4.2 (a) is contractible. Also, by Lemma 2.5 (b), each term $(\underline{CY}, \underline{Y})^{\partial\sigma_i}$ in Theorem 4.2 (a) and (b) is homotopy equivalent to a simply-connected sphere.

Proof of Theorem 1.1. Theorem 4.2 (b) implies that if K has two minimal missing faces with non-trivial intersection then a wedge of two simply-connected spheres retracts off $\Omega(\underline{X}, \underline{A})^K$. The Hilton-Milnor Theorem shows that a wedge of two such spheres is hyperbolic, and Neisendorfer and Selick [16] showed that a wedge of two such spheres has no exponent at any prime p . Hence Moore's conjecture holds in this case. On the other hand, if all the minimal missing faces of K are mutually disjoint then Theorem 4.2 (a) implies that $\Omega(\underline{X}, \underline{A})^K$ is homotopy equivalent to a finite product of spheres. This is elliptic, and as each sphere has an exponent at every prime p , so does a finite product of them. Hence Moore's Conjecture holds in this case as well. \square

Proof of Theorem 1.2. Recall that if Y is any space then ΣY is rationally homotopy equivalent to a wedge of spheres. In particular, if $\partial\sigma \subseteq K$ and each Y_i is rationally nontrivial then by Lemma 2.5 (b) the space $(\underline{CY}, \underline{Y})^{\partial\sigma}$ is rationally homotopy equivalent to a wedge of simply-connected spheres. Thus if v is a vertex of $\partial\sigma$ and $\text{rank}(\pi_*(Y_v) \otimes \mathbb{Q}) \geq 2$ then $(\underline{CY}, \underline{Y})^{\partial\sigma}$ is rationally homotopy equivalent to a wedge of at least two simply-connected spheres.

Suppose that $(\underline{X}, \underline{A})^K$ is elliptic. The homotopy decomposition $\Omega(\underline{X}, \underline{A})^K \simeq (\prod_{i=1}^m \Omega X_i) \times \Omega(\underline{CY}, \underline{Y})^K$ in Theorem 2.3 then immediately implies that each X_i must be elliptic, so condition (i) holds. This homotopy decomposition also implies that $(\underline{CY}, \underline{Y})^K$ is elliptic. Let $\sigma_1, \dots, \sigma_n$ be the minimal missing faces of K . If two of these minimal missing faces intersect, say σ_1 and σ_2 , then Theorem 4.2 implies that $\Omega((\underline{CY}, \underline{Y})^{\partial\sigma_1} \vee (\underline{CY}, \underline{Y})^{\partial\sigma_2})$ retracts off $\Omega(\underline{CY}, \underline{Y})^K$. Since each of $(\underline{CY}, \underline{Y})^{\partial\sigma_1}$ and $(\underline{CY}, \underline{Y})^{\partial\sigma_2}$ is rationally homotopy equivalent to a wedge of simply-connected spheres, the space $(\underline{CY}, \underline{Y})^{\partial\sigma_1} \vee (\underline{CY}, \underline{Y})^{\partial\sigma_2}$ is rationally homotopy equivalent to a wedge of at least two simply-connected spheres, implying that it is hyperbolic. Therefore $(\underline{CY}, \underline{Y})^K$ is hyperbolic, a contradiction. Hence the minimal missing faces of K must be mutually disjoint, implying that condition (ii) holds. Because condition (ii) holds, Theorem 4.2 implies that $\Omega(\underline{CY}, \underline{Y})^K \simeq \prod_{j=1}^n \Omega(\underline{CY}, \underline{Y})^{\partial\sigma_j}$. It has already been observed that if v is a vertex of $\partial\sigma_j$ and $\text{rank}(\pi_*(X_v) \otimes \mathbb{Q}) \geq 2$ then $(\underline{CY}, \underline{Y})^{\partial\sigma_j}$ is rationally homotopy equivalent to a wedge of at least two simply-connected spheres, and so is hyperbolic, implying that $(\underline{CY}, \underline{Y})^K$ is hyperbolic, a contradiction. Thus condition (iii) holds.

Conversely, suppose that conditions (i) to (iii) hold. By Theorem 4.2, condition (ii) implies that $\Omega(\underline{X}, \underline{A})^K \simeq (\prod_{i=1}^m \Omega X_i) \times (\prod_{j=1}^n \Omega(\underline{CY}, \underline{Y})^{\partial\sigma_j})$, where $\sigma_1, \dots, \sigma_n$ are the minimal missing faces of K . For each vertex v of any σ_i , condition (iii) states that Y_v is rationally homotopy equivalent to a sphere. Therefore Lemma 2.5 (b) implies that $(\underline{CY}, \underline{Y})^{\partial\sigma_i}$ is rationally homotopy equivalent to a sphere. As each X_i is elliptic by condition (i), it has finitely many rational homotopy groups.

Hence the homotopy decomposition for $\Omega(\underline{X}, \underline{A})^K$ implies that $(\underline{X}, \underline{A})^K$ has finitely many rational homotopy groups and so is elliptic. \square

REFERENCES

1. A. Bahri, M. Bendersky, F.R. Cohen, and S. Gilter, The polyhedral product functor: a method of decomposition for moment-angle complexes, arrangements and related spaces, *Adv. Math.* **225** (2010), 1634-1668.
2. A. Bahri, M. Bendersky, F.R. Cohen, and S. Gilter, On the rational type of moment-angle complexes, *Proc. Steklov Inst. Math.* **286** (2014), 219-223.
3. V.M. Buchstaber and T.E. Panov, *Toric topology*, Mathematical Surveys and Monographs **204**, American Mathematical Society, 2015.
4. W. Chachólski, W. Pitsch, J. Scherer and D. Stanley, Homotopy exponents for large H -spaces, *Int. Math. Res. Not. IMRN 2008*, **16**, Art. ID rnn061, 5pp.
5. M.W. Davis and T. Januszkiewicz, Convex polytopes, Coxeter orbifolds and torus actions, *Duke Math. J.* **62** (1991), 417-452.
6. G. Denham and A. Suciu, Moment-angle complexes, monomial ideals and Massey products, *Pure Appl. Math Q.* **3** (2007), 25-60.
7. Y. Félix, S. Halperin and J.-C. Thomas, Elliptic spaces II. *Enseign. Math. (2)* **39** (1993), 25-32.
8. Y. Félix and D. Tanré, Rational homotopy of the polyhedral product functor, *Proc. Amer. Math. Soc.* **137** (2009), 891-898.
9. J. Grbić and S. Theriault, The homotopy type of the complement of a coordinate subspace arrangement, *Topology* **46** (2007), 357-396.
10. J. Grbić and S. Theriault, The homotopy type of the polyhedral product for shifted complexes, *Adv. Math.* **245** (2013), 690-715.
11. K. Iriye and D. Kishimoto, Decompositions of polyhedral products, *Adv. Math.* **245** (2013), 716-736.
12. I.M. James, The suspension triad of a sphere, *Ann. of Math.* **63** (1956), 407-429.
13. J. Long, Thesis, Princeton University, 1978.
14. C.A. McGibbon and C.W. Wilkerson, Loop spaces of finite complexes at large primes, *Proc. Amer. Math. Soc.* **96** (1986), 698-702.
15. J.A. Neisendorfer, The exponent of a Moore space. *Algebraic topology and algebraic K-theory (Princeton, N.J., 1983)*, 35-71, Ann. of Math. Stud. **113**, Princeton Univ. Press, Princeton, NJ, 1987.
16. J. Neisendorfer and P. Selick, Some examples of spaces with and without homotopy exponents, *Current trends in algebraic topology, Part 1 (London, Ont., 1981)*, pp. 343-357, CMS Conf. Proc. **2**, Amer. Math. Soc., Providence, R.I., 1982.
17. G.J. Porter, The homotopy groups of wedges of suspensions, *Amer. J. Math.* **88** (1966), 655-663.
18. P. Selick, On conjectures of Moore and Serre in the case of torsion-free suspensions, *Math. Proc. Cambridge Philos. Soc.* **94** (1983), 53-60.
19. M. Stelzer, Hyperbolic spaces at large primes and a conjecture of Moore, *Topology* **43** (2004), 667-675.
20. H. Toda, On the double suspension E^2 , *J. Inst. Polytech. Osaka City Univ., Ser. A* **7** (1956), 103-145.

SCHOOL OF MATHEMATICAL SCIENCES AND LPMC, NANKAI UNIVERSITY, TIANJIN 300071, P.R. CHINA

E-mail address: `haoyanlong13@mail.nankai.edu.cn`

SCHOOL OF MATHEMATICAL SCIENCES AND LPMC, NANKAI UNIVERSITY, TIANJIN 300071, P.R. CHINA

E-mail address: `qwsun13@mail.nankai.edu.cn`

MATHEMATICAL SCIENCES, UNIVERSITY OF SOUTHAMPTON, SOUTHAMPTON SO17 1BJ, UNITED KINGDOM

E-mail address: `S.D.Theriault@soton.ac.uk`